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DIFFUSION-WAVE EQUATIONS IN NON-CANONICAL REGIONS

ABSTRACT
of the dissertation for the degree of Doctor of Philosophy (PhD) in the educational program 8D05401-Mathematics

Relevance of the topic. The classical foundation of the study is formed by works on parabolic equations and problems with changing domain geometry. M. Gevrey's work laid the basis for the analytical study of parabolic-type equations; later, the general theory was systematized by A. Friedman, free- and moving-boundary problems were investigated in detail by J. Crank, and the one-dimensional theory of heat conduction and boundary value problems was developed by J. R. Cannon. At the same time, the transition to domains with variable geometry naturally increases the role of integral-operator methods. In this regard, the article "An integral equation" by G. H. Hardy and E. C. Titchmarsh, the monograph by F. G. Tricomi, the fundamental work of Gripenberg, Londen, and Staffans, the monograph by T. A. Burton, and studies of weakly singular integral-algebraic and integro-differential equations are of fundamental importance.

In the study of boundary value problems for differential equations of fractional order, the works of A. N. Gerasimov and M. Caputo, as well as studies by M. M. Dzhrbashyan, play an essential role. A systematic exposition of the theory of fractional integrals and derivatives, and subsequently of fractional differential equations, is given in the monographs by S. G. Samko, A. A. Kilbas, O. I. Marichev, I. Podlubny, H. M. Srivastava and J. J. Trujillo, K. Diethelm, as well as in the collective volume edited by R. Hilfer. The physical and mathematical motivation for these studies is associated with the modeling of processes with memory, heredity, and anomalous transport; in this direction, the works of R. R. Nigmatullin, R. Metzler, and J. Klafter should be noted. These works contributed to the understanding of fractional equations as a natural tool for describing slow diffusion and relaxation processes.

For the analytical study of fractional diffusion and diffusion-wave equations, the apparatus of special functions, above all the Wright and Mittag-Leffler functions, is decisive. Its origins go back to the works of E. M. Wright. Later, the properties of the Wright function, its relationship with fundamental solutions of fractional-order equations, and its scaling invariance were studied in detail by R. Gorenflo, Yu. Luchko, and F. Mainardi. Also important are F. Mainardi's results on fundamental solutions of the fractional diffusion-wave equation and his monographic exposition of the theory of waves and relaxation in media with memory.

The theory of fractional diffusion and diffusion-wave equations itself developed from the first models of fractional diffusion and the corresponding fundamental solutions toward boundary value problems in bounded domains, maximum principles, and initial-boundary value formulations. The works of W. R. Schneider and W. Wyss, A. N. Kochubei, and S. D. Eidelman showed that fractional order in time is not merely a formal modification of the classical heat equation, but leads to a different structure of the fundamental solution and to new questions of existence. Subsequently, O. P. Agrawal, Yu. Luchko, K. Sakamoto, and M. Yamamoto investigated boundary and initial-boundary value problems and established results on existence, uniqueness, and properties of solutions. However, this theory is developed to a significantly lesser extent precisely for noncanonical and noncylindrical domains.

We also note the works of A. V. Pskhu, in which the following were obtained: the solution of a boundary value problem for a fractional equation, the construction of solutions by the Green's function method, a monograph on partial differential equations of fractional order, the fundamental solution of a fractional diffusion-wave equation, and the first boundary value problem in a noncylindrical domain. A special place is occupied by the work on Green's functions for multidimensional domains, since it is directly related to the subject of the dissertation, where the central objects are precisely Green's functions and integral representations in domains of complex geometry.

We also note the works of A. Kubica, P. Rybka, and K. Ryszewska on weak solutions in noncylindrical domains, as well as the works of A. V. Pskhu, M. I. Ramazanov, N. K. Gulmanov, and S. A. Iskakov devoted to boundary value problems in noncanonical domains.

The relevance of the dissertation research is determined by the need for further development of the theory of boundary value problems for fractional diffusion and diffusion-wave equations in noncanonical domains. The most significant here are three interrelated circumstances: the time-nonlocal character of fractional operators, the complex geometry of the solution domain, and the resulting need to construct new Green's functions and fractional-diffusion potentials and to investigate singular Volterra integral equations of the second kind. This set of issues constitutes the main content of the dissertation.

Purpose of research.

The purpose of the dissertation is to develop analytical methods for studying fractional diffusion and diffusion-wave equations in noncanonical domains based on the construction of fundamental solutions and Green's functions for the Riemann-Liouville and Gerasimov-Caputo fractional operators, the derivation of explicit integral representations of solutions, the reduction of initial-boundary value problems to Volterra integral equations of the second kind, and the establishment of conditions for the existence and uniqueness of regular, classical, and generalized solutions in degenerating domains with moving, narrowing, and expanding boundaries.

Research objectives:

1. To systematize and refine the auxiliary apparatus required for studying the problems under consideration, including the properties of generalized functions, Mittag-Leffler and Wright functions, and the Riemann-Liouville and Gerasimov-Caputo fractional operators.
2. To obtain an integral representation of the solution of a linear ordinary differential equation with a right-sided Liouville fractional derivative and to investigate the existence and uniqueness of a regular solution.
3. To construct fundamental solutions and Green's functions for Riemann-Liouville fractional operators in model domains, in particular on a half-line, in a half-plane, and in domains that can be reduced to canonical form by a change of variables.
4. To investigate a boundary value problem for the fractional diffusion equation with infinite memory in a narrowing noncylindrical domain, obtain an explicit integral formula for a regular solution, and reduce the problem to a Volterra integral equation of the second kind for an unknown boundary density.
5. To investigate the Dirichlet problem for a fractional diffusion equation with a right-sided Liouville derivative in an angular domain, establish jump formulas for the corresponding potential, and prove the unique solvability of the resulting Volterra integral equation.
6. To develop a weighted functional scheme for studying a diffusion-wave boundary value problem in a domain with a moving diagonal boundary, establish conditions on the weight parameters, prove theorems on the uniqueness and existence of classical and generalized solutions, and construct the Green's function of the problem under study.
7. To investigate integral transforms with the Wright kernel, obtain commutation formulas for the Stankovic transform and the right-sided Riemann-Liouville operator, and establish a criterion for the existence of exponential solutions of the corresponding fractional equation.

General research methods.

The dissertation uses methods of fractional calculus, the theory of generalized functions, and methods from the theory of partial differential equations and integral equations. In constructing solutions, Green's functions, fundamental solutions, fractional-diffusion potentials, the reflection method, changes of variables that reduce the domain to a more convenient form, and integral transform methods are applied, including the Laplace transform, the sine transform, and the Stankovic transform. Special functions play an essential role, primarily the Mittag-Leffler and

Wright functions, as well as Shilov-Gelfand type kernels. To study the solvability of boundary value problems, methods of reduction to Volterra integral equations of the second kind with weak or singular kernels, the method of successive approximations, estimates of integral operators, and methods of functional analysis are used, including work in weighted spaces, proofs of compactness of embeddings, and energy identities. The proof of existence of the constructed solutions is based on direct substitution into the original equations and boundary conditions, as well as on limiting transitions justified by convergence conditions for the corresponding integrals.

Scientific novelty.

The dissertation presents new results in the theory of fractional diffusion and diffusion-wave equations in noncanonical domains.

1. A solution of a differential equation with a right-sided Liouville fractional derivative has been found using the Green's function method; conditions for the existence and uniqueness of a regular solution have been established.
2. Fundamental solutions and Green's functions have been constructed for a number of Riemann-Liouville fractional operators in model domains; solutions of the corresponding boundary value problems have been found.
3. For the fractional diffusion equation with infinite memory in a narrowing noncylindrical domain, an explicit formula for a regular solution has been obtained, and it has been proved that the problem is reduced to a Volterra integral equation of the second kind with respect to an unknown boundary density.
4. For the Dirichlet problem in a degenerate angular domain, a representation of the solution in terms of fractional potentials has been obtained, the jump formula for the fractional double-layer potential has been derived, and the unique solvability of the corresponding Volterra integral equation has been proved.
5. For the diffusion-wave boundary value problem in a domain with a moving diagonal boundary, a weighted approach has been developed that makes it possible to establish conditions on the weight parameters, prove the compactness of the corresponding embedding, construct the Green's function, and prove uniqueness and existence theorems for classical and generalized solutions.
6. Integral transforms with the Wright kernel have been studied; commutation relations have been established for the Stankovic transform and the right-sided Riemann-Liouville operator, and a sectorial criterion for existence and the structure of exponential solutions of the corresponding fractional equation have been obtained.

Theoretical and practical significance of the research.

The theoretical significance of the dissertation lies in the further development of the theory of fractional differential equations and boundary value problems for equations of mathematical physics in noncanonical domains. The obtained results extend the Green's function method to equations with time-fractional derivatives, clarify the relationship between boundary value problems and Volterra integral equations of the second kind, and develop the analytical apparatus for studying problems in degenerating domains with moving boundaries. The development of a weighted approach to diffusion-wave problems in domains with a diagonal moving boundary, as well as the study of integral transforms with the Wright kernel for right-sided fractional operators, has substantial theoretical significance.

The practical significance of the work lies in the fact that the constructed Green's functions, explicit integral formulas for solutions, obtained estimates, and solvability criteria can be used in the study of direct and inverse problems for differential equations of fractional order, in the development of computational methods for problems with memory and moving boundaries, and in the mathematical modeling of anomalous diffusion, relaxation, and transport processes in media with complex geometry. The dissertation results can be used in research work and in the preparation of special courses on fractional calculus, equations of mathematical physics, and integral equations.

Main provisions submitted for defense.

1. The solution of a linear differential equation with a right-sided Liouville fractional derivative and theorems on the existence and uniqueness of a regular solution.
2. The construction of fundamental solutions and Green's functions for Riemann-Liouville fractional operators in model domains, as well as the corresponding integral representations of solutions to boundary value problems.
3. The existence of a regular solution to a boundary value problem for the fractional diffusion equation with infinite memory in a narrowing noncylindrical domain and the reduction of the problem to a Volterra integral equation of the second kind.
4. The existence and uniqueness of a solution to the Dirichlet problem for a fractional diffusion equation in a degenerate angular domain, the jump formula for the fractional double-layer potential, and the unique solvability of the corresponding Volterra integral equation of the second kind.
5. The weighted functional formulation of a diffusion-wave boundary value problem in a domain with a moving diagonal boundary, the Green's function of the problem, and theorems on the existence and uniqueness of classical and generalized solutions.
6. Commutation formulas for the Stankovic transform and the right-sided Riemann-Liouville operator, as well as a sectorial criterion for the existence of exponential solutions of a fractional equation with a right-sided derivative.

Reliability and validity of the conducted research.

The reliability and validity of the results obtained in the dissertation are ensured by the rigorous mathematical formulation of the problems considered, the use of established methods of fractional calculus, the theory of special functions, the theory of integral transforms and integral equations, and the completeness and logical consistency of the proofs. All main results were obtained analytically and are accompanied by a consistent justification of the admissibility of the transformations used, including interchanging integration and differentiation, limiting transitions, and the study of convergence of integrals. The constructed Green's functions and integral representations of solutions are verified by direct substitution into the original equations with account of the boundary conditions. Uniqueness theorems are based on energy identities, properties of the corresponding functional classes, and the unique solvability of Volterra integral equations. Existence theorems are constructive in nature and are based on explicit formulas, potentials, and successive approximations. Additional confirmation of reliability is provided by the consistency of the obtained results with known provisions of the general theory of fractional equations and their natural limiting cases.

Publications.

The main results of the dissertation have been published in works [49-53] indexed in the Web of Science Core Collection and Scopus databases.

1. Omarov M.T., A.V. Pskhu, Ramazanov M.I. The first boundary value problem for the fractional diffusion equation in a degenerate angular domain. Bulletin of the Karaganda University. Mathematics Series. 2024. No. 1(113). pp. 162-173. https://doi.org/10.31489/2024M1/162-173, (Q2)
2. Ramazanov, M.I., Gulmanov, N.K., Kopbalina, S.S., Omarov M.T. Solution of a Singular Integral Equation of Volterra Type of the Second Kind. Lobachevskii J Math 45, 5898-5906 (2024). https://doi.org/10.1134/S1995080224606830. (Q2)
3. Ramazanov M.I., Gulmanov N.K., Omarov M.T. On Solving a Singular Volterra Integral Equation. Filomat 39:11 (2025), 3647-3656. https://doi.org/10.2298/FIL2511647R (Q2)
4. A.D. Akhmetshin, M.T. Omarov, R.Z. Toleukhanova. A Boundary Value Problem for a Time-Fractional Diffusion Equation in a Non-Cylindrical Shrinking Domain. Bulletin of the Karaganda University. Mathematics Series, No. 1(121), 2026, pp. 37-54. https://doi.org/10.31489/2026M1/37-54.
5. M. T. Omarov, M. I. Ramazanov. Boundary Value Problem for the Fractional Diffusion Equation with the Right-Sided Liouville Operator in a Triangular Domain. Lobachevskii Journal of Mathematics, 2026, Vol. 47, No. 2, pp. 625-635

Conference abstracts:

1. Omarov M.T., A.V. Pskhu, Ramazanov M.I. An initial boundary value problem for the fractional diffusion equation in a degenerating angular domain. Traditional International April Mathematical Conference in Honor of the Day of Science of the Republic of Kazakhstan. Abstracts. Almaty: Institute of Mathematics and Mathematical Modeling. 2024. p. 177;
2. Omarov M.T., Pskhu A.V., Ramazanov M.I. The initial boundary value problem for fractional diffusion equations within a degenerate angular domain. International Scientific Conference "Mathematics in the Constellation of Sciences," dedicated to the 85th anniversary of the birth of Academician of the Russian Academy of Sciences Viktor Antonovich Sadovnichy;
3. Gulmanov N.K., Omarov M.T., Tanin A.O. Solution of a Singular Integral Equation of Volterra Type for Heat Conduction Problems. International Scientific Conference Actual Problems of Applied Mathematics and Information Technologies - Al-Khwarizmi 2024;
4. Ramazanov M.I., Gulmanov N.K., Omarov M.T. On parabolic problems in degenerating domains. International Scientific Conference "Equations of Mixed Type and Related Problems of Modern Analysis," March 11-14, 2025, Nalchik, Kabardino-Balkarian Republic, Russia;
5. Omarov M.T., Ramazanov M.I., Tanin A.O., Shayakhmetova B.K. Application of Neural Networks to Solving Inverse Problems Related to Fractional Differential Equations. Proceedings of the Eurasian International Scientific Conference "Artificial Intelligence and Inverse Problems in Science, Engineering, and Industry," April 14-16, 2025;
6. M.T. Omarov, A.V. Pskhu, M.I. Ramazanov. Solution of a Boundary Value Problem for the Fractional Diffusion Equation with a Right-Sided Liouville Operator in a Triangular Domain. Traditional International April Mathematical Conference in Honor of the Day of Science of the Republic of Kazakhstan. Abstracts. Almaty: Institute of Mathematics and Mathematical Modeling, 2025. 301 p.;
7. Gulmanov N.K., Omarov M.T., Tanin A.O. Solution of a Singular Integral Equation of Volterra Type for Heat Conduction Problems. Fundamental Science and Priorities of the 21st Century: Proceedings of the International Scientific and Practical Conference (November 29, 2024). Astana: Kazakhstan Branch of Lomonosov Moscow State University, 2025. 747 p.;
8. M.T. Omarov, A.V. Pskhu, M.I. Ramazanov. On A Linear Differential Equation With A Right-Sided Liouville-Weyl Fractional Derivative. International Conference "Current Problems of Analysis, Differential Equations, and Algebra" (EMJ-2025);
9. Omarov M.T., Pskhu A.V., Ramazanov M.I. Fractional diffusion with a right-sided Liouville derivative: a boundary value problem in a triangular domain.

International Conference on Mathematics, Mechanics, Information Technologies, and Artificial Intelligence (ICMM&IT 2025). September 24-26, 2025. Almaty, Kazakhstan;

10. Omarov M.T. Boundary value problem for the fractional diffusion equation with a right-sided Liouville operator in a triangular domain. 15th ISAAC Congress, July 21 - July 25, 2025. Venue: Nazarbayev University, Astana, Kazakhstan;

11. M.I. Ramazanov, M.T. Omarov. Boundary value problems for the heat equation in degenerating domains and Volterra equations. XVII International Kazan School-Conference "Function Theory, Its Applications, and Related Issues" (Kazan, August 23-28, 2025);

12. Omarov M.T., Ramazanov M. A boundary value problem for a time-fractional diffusion equation in a non-cylindrical shrinking domain. International Scientific Conference "Trends in Analysis and Differential Equations." Dedicated to the 65th anniversary of Professor Batirkhan Turmetov. Turkistan.

Approbation of the obtained results.

The main results of the dissertation were reported and discussed at the above-mentioned conferences and seminars.

Doctoral candidate's contribution to each publication.

In the five main works written in co-authorship with the scientific consultants and co-authors, the scientific consultants formulated the problem statement and determined the research methodology, while the doctoral candidate independently formulated the main and auxiliary results and proved them.

Structure and scope of the dissertation.

The dissertation comprises 144 pages and consists of an introduction, four sections, a conclusion, and a list of references. The numbering of mathematical statements (theorems, lemmas, remarks) and formulas is three-level: the first digit denotes the chapter number, the second denotes the section number, and the third denotes the sequential number of the mathematical statement or formula itself.

Number of sources used: 73.

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